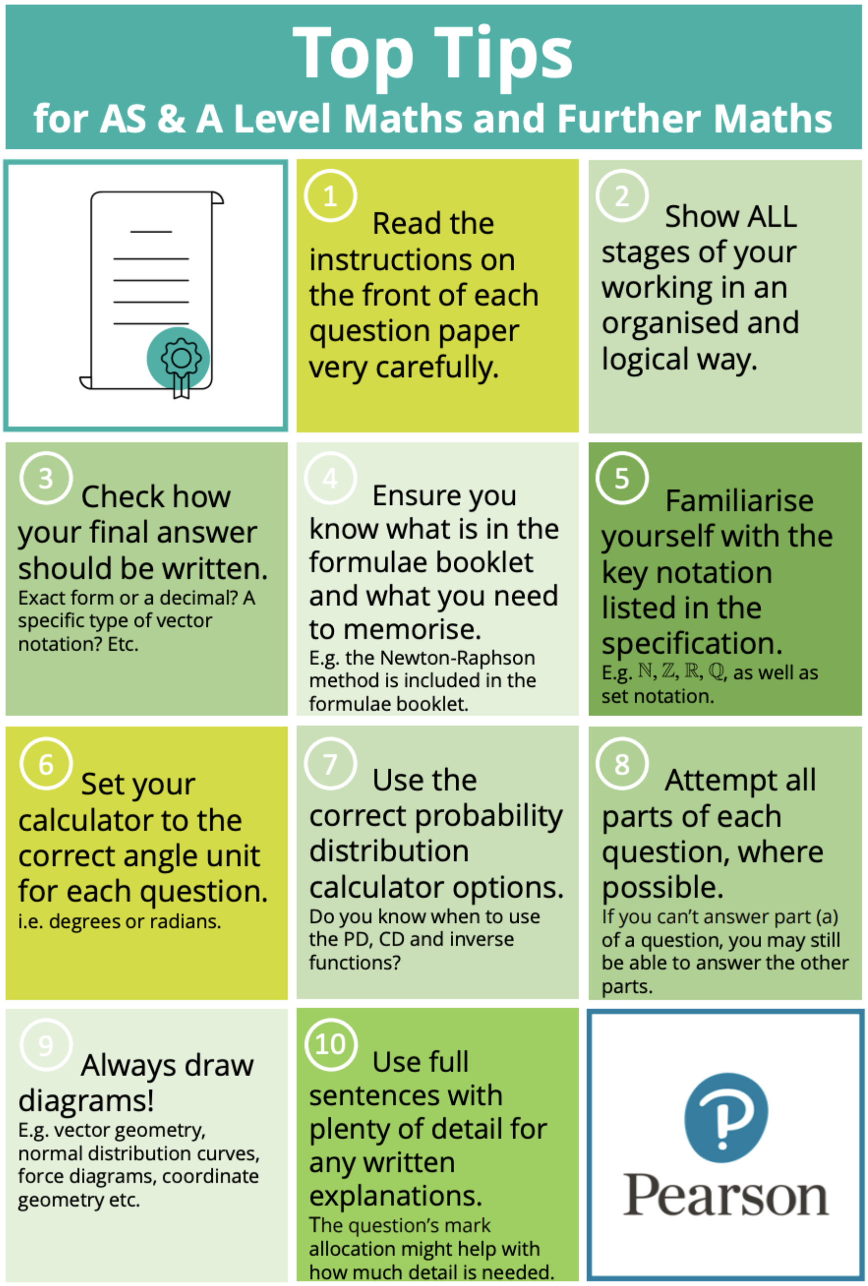
Revision lesson on differential equations

you CHAN do it



Question:	1	2	3	4	5	6
Points:	5	5	8	13	5	9
Score:						

Question:	7	8	9	10		Total
Points:	15	15	12	14		101
Score:						

Working Space

1.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$$

(a) find a general solution in the form $y^2 = f(x)$

(3)

(b) find a particular solution, given that when $x = 1$, $y = 2$

(2)

$\int x \, dx = \int y \, dy$

$\frac{x^2}{2} = \frac{y^2}{2} + c$

$y^2 = x^2 + 2c$

b) $4 = 1 + 2c$

$3 = 2c$

$c = \frac{3}{2}$

$y^2 = x^2 + 3$

2.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$$

(a) find a general solution in the form $y = f(x)$

(3)

(b) find a particular solution, given that when $x = 1$, $y = 2$

(2)

$\int \frac{1}{y} \, dy = \int \frac{1}{x} \, dx$

$\ln|y| = \ln|x| + c$

$y = Ax$

$A = 2$ $\boxed{y = 2x}$

3. (a) Find

$$\int \frac{9x+6}{x}\,\mathrm{d}x, \quad x>0$$

(2)

(b) Given that $y=8$ at $x=1$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(9x+6)y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2=g(x)$.

(6)

a) $\int 9+\frac{6}{x}\,dx = 9x+6\ln|x| + c$

b) $\int \frac{9x+6}{x}\,dx = \int y^{\frac{-1}{3}}\,dy$

$9x+6\ln x = \frac{3}{2}y^{2/3}+c$

$9=6+c$

$c=3$

$9x+6\ln x=\frac{3}{2}y^{2/3}+3$

$y^{2/3}=6x+4\ln x-2$

$y^2=(6x+4\ln x-2)^3$

Question Number	Scheme		Marks	
	(a) $\int \frac{9x+6}{x}\,dx=\int\left(9+\frac{6}{x}\right)dx$		M1	
	$=9x+6\ln x\ (+C)$		A1	(2)
	(b) $\int \frac{1}{y^{\frac{1}{3}}}\,dy=\int\frac{9x+6}{x}\,dx$	Integral signs not necessary	B1	
	$\int y^{-\frac{1}{3}}\,dy=\int\frac{9x+6}{x}\,dx$			
	$\frac{y^{\frac{2}{3}}}{\frac{2}{3}}=9x+6\ln x\ (+C)$	$\pm ky^{\frac{2}{3}}=$ their (a)	M1	
	$\frac{3}{2}y^{\frac{2}{3}}=9x+6\ln x\ (+C)$	ft their (a)	A1ft	
	$y=8,\ x=1$			
	$\frac{3}{2}8^{\frac{2}{3}}=9+6\ln 1+C$		M1	
	$C=-3$		A1	
	$y^{\frac{2}{3}}=\frac{2}{3}(9x+6\ln x-3)$			
	$y^2=(6x+4\ln x-2)^3\quad\left(=8(3x+2\ln x-1)^3\right)$		A1	(6)
				[8]

4. Liquid is pouring into a container at a constant rate of 20 $\text{cm}^3\ \text{s}^{-1}$ and is leaking out at a rate proportional to the volume of liquid already in the container.

(a) Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 20 - kV,$$

where k is a positive constant.

(2)

The container is initially empty.

(b) By solving the differential equation, show that

$$V = A + B\mathrm{e}^{-kt},$$

giving the values of A and B in terms of k.

(6)

Given also that $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$ when $t = 5$,

(c) find the volume of liquid in the container at 10 s after the start.

(5)

(a) $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the rate of increase of volume (with respect to time) B1

$-kV$: k is constant of proportionality and the negative shows decrease (or loss) giving $\frac{\mathrm{d}V}{\mathrm{d}t} = 20 - kV$ (*) B1 2

These Bs are to be awarded independently

(b) $\int \frac{1}{20 - kV}\,\mathrm{d}V = \int 1\,\mathrm{d}t$ M1

separating variables

$-\frac{1}{k}\ln(20 - kV) = t\ (+C)$ M1 A1

Using $V = 0$, $t = 0$ to evaluate the constant of integration M1

$c = -\frac{1}{k}\ln 20$

$t = \frac{1}{k}\ln\left(\frac{20}{20 - kV}\right)$

Obtaining answer in the form $V = A + B\,\mathrm{e}^{-kt}$ M1

$V = \frac{20}{k} - \frac{20}{k}\mathrm{e}^{-kt}$ A1 6

Accept $\frac{20}{k}(1 - \mathrm{e}^{-kt})$

Alternative to (b)

Using printed answer and differentiating $\frac{\mathrm{d}V}{\mathrm{d}t} = -kB\,\mathrm{e}^{-kt}$ M1

Substituting into differential equation $-kB\,\mathrm{e}^{-kt} = 20 - kA - kB\,\mathrm{e}^{-kt}$ M1

$A = \frac{20}{k}$ M1 A1

Using $V = 0$, $t = 0$ in printed answer to obtain $A + B = 0$ M1

$B = -\frac{20}{k}$ A1 6

Working Space

5. Water is flowing into a large container and is leaking from a hole at the base of the container.

At time t seconds after the water starts to flow, the volume, V cm^3, of water in the container is modelled by the differential equation

$$\frac{dV}{dt} = 300 - kV$$

where k is a constant.

(a) Solve the differential equation to show that, according to the model,

$$V = \frac{300}{k} + Ae^{-kt}$$

where A is a constant.

(5)

Skipping Subsequent Parts in the original question this time, but well done on getting this far, year 13s!

Jan 2022 (unused)

Bonus Q:

7. Water is flowing into a large container and is leaking from a hole at the base of the container.

At time t seconds after the water starts to flow, the volume, V cm^3, of water in the container is modelled by the differential equation

$$\frac{dV}{dt} = 300 - kV$$

where k is a constant.

"you cannot add/subtract" you can only "multiply"

(a) Solve the differential equation to show that, according to the model,

$$V = \frac{300}{k} + Ae^{-kt}$$

where A is a constant.

(5)

a) $\left(\frac{1}{-k}\right)\int \frac{1(-k)}{(300-kV)}\, dV = \int 1 \, dt$

$\left(\frac{1}{-k}\right) \ln|300-kV| = t + c$

$\ln|300-kV| = -kt - kc$

$e^{\ln(300-kV)} = e^{-kt + \ln\lambda}$

Note kc is a constant, so could be just "$\ln\lambda$"

$300 - kV = \lambda e^{-kt}$

$kV = 300 - \lambda e^{-kt}$

"avoid using "A" until last step"

$V = \frac{300}{k} - \frac{\lambda}{k} e^{-kt}$

$V = \frac{300}{k} + Ae^{-kt}$ //

6. A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3°C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ°C.

The rate of change of the temperature of the water in the bottle is modelled by the differential equation,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{(3-\theta)}{125}$$

(a) By solving the differential equation, show that,

$$\theta = A\mathrm{e}^{-0.008t} + 3$$

where A is a constant.

(4)

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16°C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10°C, giving your answer to the nearest minute.

(5)

Integration, Differential equations

Question | Mark Scheme | Examiner's Report

Candidates found this a challenging question. Some candidates did not attempt to separate the variables in (a). They were also not able to deal with the context of the question in (b).

In (a), those candidates who were able to separate the variables, were usually able to integrate both sides correctly, although a number made a sign error by integrating $\frac{1}{3-\theta}$ to obtain $\ln|3-\theta|$. A significant number of candidates omitted the constant of integration " $+ c$ " and so were not able to gain the final mark. A significant number of candidates did not show sufficient steps in order to progress from $-\ln|3-\theta| = \frac{1}{125}t + c$ to the result $\theta = A\mathrm{e}^{-0.008t} + 3$. Common errors included candidates removing their logarithms incorrectly to give an equation of the form $3 - \theta = \mathrm{e}^{-\frac{1}{125}t} + A$ or candidates stating the constant A as -1.

(b) was often better answered with some candidates scoring no marks in (a) and full marks in (b). Those candidates who used $\theta = A\mathrm{e}^{-0.008t} + 3$ were more successful in this part. They were usually able to write down the condition $\theta = 16$ when $t = 0$ in order to find $A = 13$. Some candidates misinterpreted the context of the question to write down the condition $\theta = 6$ when $t = 0$, yielding the result of $A = 3$. Other incorrect values of A seen by examiners included -1, 16 or 1. Many candidates who found A correctly were usually able to substitute $\theta = 10$ into $\theta = 16\mathrm{e}^{-0.008t} + 3$ and manipulate the result correctly in order to find the correct time.

Question Number	Scheme		Marks
(a)	$\left\{\frac{d\theta}{dt} = \frac{(3-\theta)}{125}\right\} \Rightarrow \int \frac{1}{3-\theta}\,d\theta = \int \frac{1}{125}\,dt$ or $\int \frac{125}{3-\theta}\,d\theta = \int dt$		B1
	$-\ln(\theta-3) = \frac{1}{125}t\ \{+c\}$ or $-\ln(3-\theta) = \frac{1}{125}t\ \{+c\}$	See notes.	M1 A1
	$\ln(\theta-3) = -\frac{1}{125}t + c$		
	$\theta - 3 = e^{-\frac{1}{125}t+c}$ or $e^{-\frac{1}{125}t}e^{c}$ $\theta = Ae^{-0.008t} + 3$ *	***Correct*** completion to $\theta = Ae^{-0.008t} + 3$.	A1
			[4]
(b)	$\{t=0,\ \theta=16 \Rightarrow\}\quad 16 = Ae^{-0.008(0)} + 3;\ \Rightarrow \underline{A = 13}$	***See notes.***	M1; A1
	$10 = 13e^{-0.008t} + 3$	Substitutes $\theta = 10$ into an equation **of the form** $\theta = Ae^{-0.008t} + 3$, or equivalent. ***See notes.***	M1
	$e^{-0.008t} = \frac{7}{13} \Rightarrow -0.008t = \ln\left(\frac{7}{13}\right)$	Correct algebra to $-0.008t = \ln k$, where k is a positive value. ***See notes.***	M1
	$\left\{t = \frac{\ln\left(\frac{7}{13}\right)}{(-0.008)}\right\} = 77.3799... = 77$ (nearest minute)	awrt 77	A1
			[5] 9

ace

(a)	**B1: (M1 on epen)** Separates variables as shown. $d\theta$ and dt should be in the correct positions, though this mark can be implied by later working. Ignore the integral signs. **M1: Both** $\pm\lambda\ln(3-\theta)$ or $\pm\lambda\ln(\theta-3)$ **and** $\pm\mu t$ where λ and μ are constants. **A1:** For $-\ln(\theta-3) = \frac{1}{125}t$ or $-\ln(3-\theta) = \frac{1}{125}t$ or $-125\ln(\theta-3) = t$ or $-125\ln(3-\theta) = t$ **Note:** $+c$ is not needed for this mark. **A1:** Correct completion to $\theta = Ae^{-0.008t} + 3$. **Note:** $+c$ is needed for this mark. **Note:** $\ln(\theta-3) = -\frac{1}{125}t + c$ leading to $\theta - 3 = e^{-\frac{1}{125}t} + e^{c}$ or $\theta - 3 = e^{-\frac{1}{125}t} + A$, would be final A0. **Note:** From $-\ln(\theta-3) = \frac{1}{125}t + c$, then $\ln(\theta-3) = -\frac{1}{125}t + c$ $\Rightarrow \theta - 3 = e^{-\frac{1}{125}t+c}$ **or** $\theta - 3 = e^{-\frac{1}{125}t}e^{c} \Rightarrow \theta = Ae^{-0.008t} + 3$ is required for A1. **Note:** From $-\ln(3-\theta) = \frac{1}{125}t + c$, then $\ln(3-\theta) = -\frac{1}{125}t + c$ $\Rightarrow 3 - \theta = e^{-\frac{1}{125}t+c}$ **or** $3 - \theta = e^{-\frac{1}{125}t}e^{c} \Rightarrow \theta = Ae^{-0.008t} + 3$ is sufficient for A1. **Note:** The jump from $3 - \theta = Ae^{-\frac{1}{125}t}$ to $\theta = Ae^{-0.008t} + 3$ is fine. **Note:** $\ln(\theta-3) = -\frac{1}{125}t + c \Rightarrow \theta - 3 = Ae^{-\frac{1}{125}t}$, where candidate writes $A = e^{c}$ is also acceptable.

(b)	**M1:** (B1 on epen) Substitutes $\theta = 16, t = 0$, into either their equation containing an unknown constant or the printed equation. **Note:** You can imply this method mark. **A1:** (M1 on epen) $A = 13$. **Note:** $\theta = 13e^{-0.008t} + 3$ without any working implies the first two marks, M1A1. **M1:** Substitutes $\theta = 10$ into an equation **of the form** $\theta = Ae^{-0.008t} + 3$, or equivalent. where A is a positive or negative numerical value and A can be equal to 1 or -1. **M1:** Uses correct algebra to rearrange **their equation** into the form $-0.008t = \ln k$, where k is ***a positive numerical value.*** **A1:** awrt 77 or awrt 1 hour 17 minutes.	
	<u>***Alternative Method 1 for part (b)***</u>	
	$\int \frac{1}{3-\theta}\,d\theta = \int \frac{1}{125}\,dt \Rightarrow -\ln(\theta-3) = \frac{1}{125}t + c$	
	$\{t=0,\ \theta=16 \Rightarrow\}\ -\ln(16-3) = \frac{1}{125}(0) + c$ $\Rightarrow c = -\ln 13$	**M1:** Substitutes $t = 0, \theta = 16$, into $-\ln(\theta-3) = \frac{1}{125}t + c$ **A1:** $c = -\ln 13$
	$-\ln(\theta-3) = \frac{1}{125}t - \ln 13$ or $\ln(\theta-3) = -\frac{1}{125}t + \ln 13$	
	$-\ln(10-3) = \frac{1}{125}t - \ln 13$	**M1:** Substitutes $\theta = 10$ into an equation **of the form** $\pm\lambda\ln(\theta-3) = \pm\frac{1}{125}t \pm \mu$ where λ, μ are numerical values.
	$\ln 13 - \ln 7 = \frac{1}{125}t$	**M1:** Uses correct algebra to rearrange **their equation** into the form $\pm 0.008t = \ln C - \ln D$, where C, D are ***positive numerical values.***
	$t = 77.3799... = 77$ (nearest minute)	**A1:** awrt 77.
	<u>***Alternative Method 2 for part (b)***</u>	
	$\int \frac{1}{3-\theta}\,d\theta = \int \frac{1}{125}\,dt \Rightarrow -\ln\lvert 3-\theta\rvert = \frac{1}{125}t + c$	
	$\{t=0,\ \theta=16 \Rightarrow\}\ -\ln\lvert 3-16\rvert = \frac{1}{125}(0) + c$ $\Rightarrow c = -\ln 13$	**M1:** Substitutes $t = 0, \theta = 16$, into $-\ln(3-\theta) = \frac{1}{125}t + c$ **A1:** $c = -\ln 13$
	$-\ln\lvert 3-\theta\rvert = \frac{1}{125}t - \ln 13$ or $\ln\lvert 3-\theta\rvert = -\frac{1}{125}t + \ln 13$	
	$-\ln(3-10) = \frac{1}{125}t - \ln 13$	**M1:** Substitutes $\theta = 10$ into an equation **of the form** $\pm\lambda\ln(3-\theta) = \pm\frac{1}{125}t \pm \mu$ where λ, μ are numerical values.
	$\ln 13 - \ln 7 = \frac{1}{125}t$	**M1:** Uses correct algebra to rearrange **their equation** into the form $\pm 0.008t = \ln C - \ln D$, where C, D are ***positive numerical values.***
	$t = 77.3799... = 77$ (nearest minute)	**A1:** awrt 77.

(b)	<u>***Alternative Method 3 for part (b)***</u>	
	$\int_{16}^{10} \frac{1}{3-\theta}\,d\theta = \int_{0}^{t} \frac{1}{125}\,dt$ $= \left[-\ln\lvert 3-\theta\rvert\right]_{16}^{10} = \left[\frac{1}{125}t\right]_{0}^{t}$	**M1A1:** $\ln 13$ **M1:** Substitutes limit of $\theta = 10$ correctly.
	$-\ln 7 - -\ln 13 = \frac{1}{125}t$	**M1:** Uses correct algebra to rearrange **their own equation** into the form $\pm 0.008t = \ln C - \ln D$, where C, D are ***positive numerical values.***
	$t = 77.3799... = 77$ (nearest minute)	**A1:** awrt 77.
	<u>***Alternative Method 4 for part (b)***</u>	
	$\{\theta = 16 \Rightarrow\}\quad 16 = Ae^{-0.008t} + 3$ $\{\theta = 10 \Rightarrow\}\quad 10 = Ae^{-0.008t} + 3$	**M1*:** Writes down a pair of equations in A and t, for $\theta = 16$ and $\theta = 10$ with either A unknown or A being a positive or negative value. **A1:** Two equations with an unknown A.
	$-0.008t = \ln\left(\frac{13}{A}\right)$ or $-0.008t = \ln\left(\frac{7}{A}\right)$	**M1:** Uses ***correct algebra*** to solve both of **their equations** leading to answers of the form $-0.008t = \ln k$, where k is ***a positive numerical value.***
	$t_{(1)} = \frac{\ln\left(\frac{13}{A}\right)}{-0.008}$ and $t_{(2)} = \frac{\ln\left(\frac{7}{A}\right)}{-0.008}$	
	$t = t_{(1)} - t_{(2)} = \frac{\ln\left(\frac{13}{A}\right)}{-0.008} - \frac{\ln\left(\frac{7}{A}\right)}{-0.008}$	**M1:** Finds difference between the two times. (either way round).
	$\left\{t = \frac{\ln\left(\frac{7}{13}\right)}{(-0.008)}\right\} = 77.3799... = 77$ (nearest minute)	**A1:** awrt 77. Correct solution only.

Spot the Difference!

C34 October 2017 QP.pdf
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n assist with filling in this form.

11. (a) Given $0 \leqslant h < 25$, use the substitution $u = 5 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{5-\sqrt{h}} = -10\ln\left(5-\sqrt{h}\right) - 2\sqrt{h} + k$$

where k is a constant.

(6)

A team of scientists is studying a species of tree.

The rate of change in height of a tree of this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.2}\left(5-\sqrt{h}\right)}{5}$$

where h is the height of the tree in metres and t is the time in years after the tree is planted.

One of these trees is 2 metres high when it is planted.

(b) Use integration to calculate the time it would take for this tree to reach a height of 15 metres, giving your answer to one decimal place.

(7)

(c) Hence calculate the rate of change in height of this tree when its height is 15 metres. Write your answer in centimetres per year to the nearest centimetre.

(1)

January 2014 (IAL) QP - C34 Edexcel 28 / 44 100%

PhysicsAndMathsTutor.com January 2014 (IAL)

Leave blank

9. (a) Use the substitution $u = 4 - \sqrt{x}$ to find

$$\int \frac{\mathrm{d}x}{4-\sqrt{x}}$$

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{4-\sqrt{h}}{20}$$

where h is the height in metres and t is the time measured in years after the tree is planted.

(b) Find the range in values of h for which the height of a tree in this species is increasing.

(2)

(c) Given that one of these trees is 1 metre high when it is planted, calculate the time it would take to reach a height of 10 metres. Write your answer to 3 significant figures.

(7)

June 2019 QP - Paper 2 Edexcel Maths A-level 44 / 48 100%

14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4 - \sqrt{h}} = -8\ln\left|4 - \sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

7.

January 2014 (IAL) QP - C34 Edexcel 28 / 44 100%

Leave blank

9. (a) Use the substitution $u = 4 - \sqrt{x}$ to find

$$\int \frac{\mathrm{d}x}{4 - \sqrt{x}}$$

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{4 - \sqrt{h}}{20}$$

where h is the height in metres and t is the time measured in years after the tree is planted.

(b) Find the range in values of h for which the height of a tree in this species is increasing.

(2)

(c) Given that one of these trees is 1 metre high when it is planted, calculate the time it would take to reach a height of 10 metres. Write your answer to 3 significant figures.

(7)

Leave blank

9. (a) Use the substitution $u = 4 - \sqrt{x}$ to find

$$\int \frac{dx}{4 - \sqrt{x}}$$

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{dh}{dt} = \frac{4 - \sqrt{h}}{20}$$

where h is the height in metres and t is the time measured in years after the tree is planted.

(b) Find the range in values of h for which the height of a tree in this species is increasing. **(2)**

(c) Given that one of these trees is 1 metre high when it is planted, calculate the time it would take to reach a height of 10 metres. Write your answer to 3 significant figures. **(7)**

Long way (Not recommended)

$$\frac{du}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{-1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{-1}{2(4-u)}$$

Easier way

$$\sqrt{x} = 4 - u$$

$$x = (4-u)^2$$

$$\frac{dx}{du} = 2(4-u)(-1)$$

$$\frac{dx}{du} = -2(4-u)$$

$$dx = (u-4)(2)\,du$$

$$\int \frac{1}{u}(u-4)(2)\,du$$

$$= 2\int 1 - \frac{4}{u}\,du = 2u - 8\ln|u| + c$$

$$= 2(4-\sqrt{x}) - 8\ln|4-\sqrt{x}| + c$$

$$= -2\sqrt{x} - 8\ln|4-\sqrt{x}| + k \text{ //}$$

$$\frac{dh}{dt} > 0 \quad 4-\sqrt{h} > 0 \quad \sqrt{h} < 4 \quad h < 16$$

$$0 < h < 16$$

$$\frac{dh}{dt} = \frac{4-\sqrt{h}}{20}$$

$$\int \frac{1}{4-\sqrt{h}}\,dh = \int \frac{1}{20}\,dt$$

$$-2\sqrt{h} - 8\ln|4-\sqrt{h}| + k = \frac{1}{20}t$$

$$t = 0 \quad = 1$$

$$-2 - 8\ln 3 + k = 0$$

$$k = 2 + 8\ln 3$$

$h = 10$

$$-2\sqrt{10} - 8\ln|4-\sqrt{10}| + 2 + 8\ln 3 = \frac{1}{20}t$$

Math Rad Norm2 d/c a+bi

$-2\sqrt{10} - 8\ln\ |4-\sqrt{10}| + 2 + 8$

5.88089155

Ans×20

117.617831

Abs | Int | Frac | Rnd | Intg

= 118 years //

11. (a) Given $0 \leqslant h < 25$, use the substitution $u = 5 - \sqrt{h}$ to show that

$$\int \frac{dh}{5-\sqrt{h}} = -10\ln\left(5-\sqrt{h}\right) - 2\sqrt{h} + k$$

where k is a constant.

Aim: replace all $h \to u$

(6)

Example (Long way)

$\frac{d}{dh}u = -\frac{1}{2}h^{-\frac{1}{2}}$

$\frac{du}{dh} = \frac{-1}{2\sqrt{h}}$

$\frac{du}{dh} = \frac{-1}{2(5-u)}$

$\frac{du}{dh} = \frac{1}{2u-10}$

Better way

$\sqrt{h} = 5-u$

$h = (5-u)^2$

$\frac{dh}{du} = 2(5-u)(-1)$

$\frac{dh}{du} = 2u-10$

$\int \frac{1}{u}(2u-10)\,du$

$\int \frac{2u-10}{u}\,du$

$= \int 2 - \frac{10}{u}\,du$

$= 2u - 10\ln u + c$

$= 2|5-\sqrt{h}| - 10\ln|5-\sqrt{h}| + c$

$= 10 - 2\sqrt{h} - 10\ln|5-\sqrt{h}| + c$

$= -10\ln|5-\sqrt{h}| - 2\sqrt{h} + k$

A team of scientists is studying a species of tree.

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$$\frac{dh}{dt} = \frac{t^{0.2}\left(5-\sqrt{h}\right)}{5}$$

where h is the height of the tree in metres and t is the time in years after the tree is planted.

One of these trees is 2 metres high when it is planted.

(b) Use integration to calculate the time it would take for this tree to reach a height of 15 metres, giving your answer to one decimal place.

(7)

$\int \frac{1}{5-\sqrt{h}}\,dh = \int \frac{t^{0.2}}{5}\,dt$ { separate the variables }

$-10\ln|5-\sqrt{h}| - 2\sqrt{h} + k = \frac{t^{1.2}}{6}$ { General solutions }

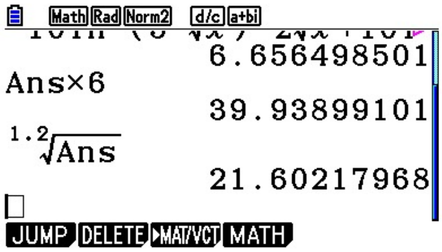
$t=0,\ h=2$

$-10\ln|5-\sqrt{2}| - 2\sqrt{2} + k = 0$

$k = 10\ln|5-\sqrt{2}| + 2\sqrt{2}$

$$-10\ln|5-\sqrt{h}|-2\sqrt{h}+10\ln|5-\sqrt{2}|+2\sqrt{2}=\frac{t^{1.2}}{6}$$

$h=15 \quad t=?$

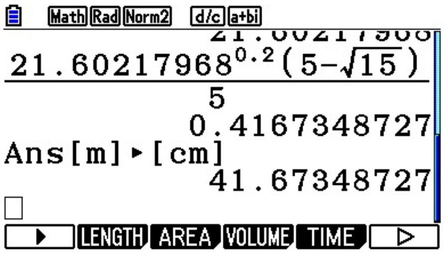


$t=21.6$

$$\frac{dh}{dt}=\frac{t^{0.2}\left(5-\sqrt{h}\right)}{5}$$

(c) Hence calculate the rate of change in height of this tree when its height is 15 metres. Write your answer in centimetres per year to the nearest centimetre.

(1)



0.4167… metre

42 centimetres per year

9. (a) Express $\dfrac{1}{P(5-P)}$ in partial fractions.

(3)

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{15}P(5-P), \quad t \geqslant 0$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0, P = 1$,

(b) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + c\mathrm{e}^{-\frac{1}{3}t}}$$

where a, b and c are integers.

(8)

(c) Hence show that the population cannot exceed 5000

(1)

Algebra, Integration, Differential equations

Question | Mark Scheme | Examiner's Report

In part (a), the majority of candidates were able to write down the correct identity to find their constants correctly, although a few candidates forgot to express their answer as a partial fraction as requested in the question.

A significant minority of candidates who completed part (a) correctly, made no attempt at part (b). Most candidates, however, were able to separate the variables, although some did this incorrectly, or did not try. The majority used their part (a) answer and integrated this to obtain an expression involving two ln terms. Although many integrated their expression correctly, some made a sign error by integrating $\frac{1}{5-P}$ to obtain $ln(5-P)$. Those candidates who integrated $\frac{1}{5P}$ and $\frac{1}{25-5P}$ to $\frac{1}{5}\ln 5P$ and $-\frac{1}{5}\ln(25-5P)$, respectively, tended to find subsequent manipulation more difficult. A significant number of candidates did not attempt to find a constant of integration - with some omitting it from their working whilst others referring to "+ c" and not attempting to use the boundary condition of $t = 0$ and $P = 1$ to find its value. Most candidates were able to apply the subtraction (or sometimes the addition) law of logarithms correctly for their expression but a number of candidates struggled to correctly remove the logarithms from their integrated equation, with incorrect manipulation of $\ln\left(\frac{P}{5-P}\right) = \frac{1}{3}t - \ln 4$ leading to $\frac{P}{5-P} = \mathrm{e}^{\frac{1}{3}t} - 4$ usually seen.

A significant number of those candidates who removed logarithms correctly were able to manipulate their result to make P the subject of their equation, although a number of these candidates could not make the final step of manipulating $P = \frac{5\mathrm{e}^{\frac{1}{3}t}}{(4+\mathrm{e}^{\frac{1}{3}t})}$ into $\frac{5}{(1+4\mathrm{e}^{-\frac{1}{3}t})}$.

Of all the 8 questions, question 8(b) was the most demanding in terms of a need for accuracy, and about 20% of candidates were able to score all 8 marks in this part.

Very few candidates gained the mark in part (c). Many were able to show that P could not be equal to 5, and some of them also looked at what happens to P as t approaches infinity, but then failed to point out that the function was strictly increasing. Few candidates were able to state $1 + 4\mathrm{e}^{-\frac{1}{3}t} > 1$ implied $P < 5$, but some of them did not go on to give a conclusion in relation to the context of the question.

Question Number	Scheme		Marks
(a)	$1 = A(5-P) + BP$	Can be implied.	M1
	$A = \frac{1}{5},\ B = \frac{1}{5}$	Either one.	A1
	giving $\frac{\frac{1}{5}}{P} + \frac{\frac{1}{5}}{(5-P)}$	See notes.	A1 **cao, aef**
			[3]
(b)	$\int \frac{1}{P(5-P)}\,\mathrm{d}P = \int \frac{1}{15}\,\mathrm{d}t$		B1
	$\frac{1}{5}\ln P - \frac{1}{5}\ln(5-P) = \frac{1}{15}t\ \ (+c)$		M1* A1ft
	$\{t=0, P=1 \Rightarrow\}\ \ \frac{1}{5}\ln 1 - \frac{1}{5}\ln(4) = 0 + c\ \ \ \left\{\Rightarrow\ c = -\frac{1}{5}\ln 4\right\}$		dM1*
	eg: $\frac{1}{5}\ln\left(\frac{P}{5-P}\right) = \frac{1}{15}t - \frac{1}{5}\ln 4$	Using any of the subtraction (or addition) laws for logarithms CORRECTLY	dM1*
	$\ln\left(\frac{4P}{5-P}\right) = \frac{1}{3}t$		
	eg: $\frac{4P}{5-P} = e^{\frac{1}{3}t}$ or eg: $\frac{5-P}{4P} = e^{-\frac{1}{3}t}$	Eliminate ln's correctly.	dM1*
	gives $4P = 5e^{\frac{1}{3}t} - Pe^{\frac{1}{3}t}\ \Rightarrow\ P(4 + e^{\frac{1}{3}t}) = 5e^{\frac{1}{3}t}$		
	$P = \frac{5e^{\frac{1}{3}t}}{(4 + e^{\frac{1}{3}t})}\ \ \left\{\frac{(\div e^{\frac{1}{3}t})}{(\div e^{\frac{1}{3}t})}\right\}$	Make P the subject.	dM1*
	$P = \frac{5}{(1 + 4e^{-\frac{1}{3}t})}$ or $P = \frac{25}{(5 + 20e^{-\frac{1}{3}t})}$ etc.		A1
			[8]
(c)	$1 + 4e^{-\frac{1}{3}t} > 1\ \Rightarrow\ P < 5$. So population cannot exceed 5000.		B1
			[1] **12**

'pace

(a)

M1: Forming a correct identity. For example, $1 = A(5-P) + BP$. **Note** A and B not referred to in question.

A1: Either one of $A = \frac{1}{5}$ or $B = \frac{1}{5}$.

A1: $\frac{\frac{1}{5}}{P} + \frac{\frac{1}{5}}{(5-P)}$ or any equivalent form, eg: $\frac{1}{5P} + \frac{1}{25-5P}$, etc. Ignore subsequent working.

This answer must be stated in part (a) only.

A1 can also be given for a candidate who finds both $A = \frac{1}{5}$ and $B = \frac{1}{5}$ and $\frac{A}{P} + \frac{B}{5-P}$ is seen in their working.

Candidate can use 'cover-up' rule to write down $\frac{\frac{1}{5}}{P} + \frac{\frac{1}{5}}{(5-P)}$, as so gain all three marks.

Candidate cannot gain the marks for part (a) in part (b).

(b)

B1: Separates variables as shown. $\mathrm{d}P$ and $\mathrm{d}t$ should be in the correct positions, though this mark can be implied by later working. Ignore the integral signs.

M1*: Both $\pm\lambda\ln P$ and $\pm\mu\ln(\pm 5 \pm P)$, where λ and μ are constants.

Or $\pm\lambda\ln mP$ and $\pm\mu\ln(n(\pm 5 \pm P))$, where λ, μ, m and n are constants.

A1ft: Correct follow through integration of both sides from their $\int \frac{\lambda}{P} + \frac{\mu}{(5-P)}\,\mathrm{d}P = \int K\,\mathrm{d}t$ with or without $+c$

dM1*: Use of $t = 0$ and $P = 1$ in an integrated equation containing c

dM1*: Using ANY of the subtraction (or addition) laws for logarithms CORRECTLY.

dM1*: Apply logarithms (or take exponentials) to eliminate ln's CORRECTLY from their equation.

dM1*: A full ACCEPTABLE method of rearranging to make P the subject. (**See below for examples!**)

A1: $P = \frac{5}{(1 + 4e^{-\frac{1}{3}t})}\ \{\text{where } a = 5, b = 1, c = 4\}$.

Also allow any "integer" multiples of this expression. For example: $P = \frac{25}{(5 + 20e^{-\frac{1}{3}t})}$

Note: If the first method mark (M1*) is not awarded then the candidate cannot gain any of the six remaining marks for this part of the question.

Note: $\int \frac{1}{P(5-P)}\,\mathrm{d}P = \int 15\,\mathrm{d}t\ \Rightarrow\ \int \frac{\frac{1}{5}}{P} + \frac{\frac{1}{5}}{(5-P)}\,\mathrm{d}P = \int 15\,\mathrm{d}t\ \Rightarrow\ \ln P - \ln(5-P) = 15t$ is B0M1A1ft.

dM1* for making P the subject

Note there are three type of manipulations here which are considered acceptable to make P the subject.

(1) M1 for $\frac{P}{5-P} = e^{\frac{1}{3}t} \Rightarrow P = 5e^{\frac{1}{3}t} - Pe^{\frac{1}{3}t}\ \Rightarrow\ P(1 + e^{\frac{1}{3}t}) = 5e^{\frac{1}{3}t} \Rightarrow P = \frac{5}{(1 + e^{-\frac{1}{3}t})}$

(2) M1 for $\frac{P}{5-P} = e^{\frac{1}{3}t} \Rightarrow \frac{5-P}{P} = e^{\frac{1}{3}t} \Rightarrow \frac{5}{P} - 1 = e^{\frac{1}{3}t} \Rightarrow\ \frac{5}{P} = e^{\frac{1}{3}t} + 1 \Rightarrow\ P = \frac{5}{(1 + e^{\frac{1}{3}t})}$

(3) M1 for $P(5-P) = 4e^{\frac{1}{3}t} \Rightarrow P^2 - 5P = -4e^{\frac{1}{3}t} \Rightarrow \left(P - \frac{5}{2}\right)^2 - \frac{25}{4} = -4e^{\frac{1}{3}t}$ leading to $P = \ldots$

Note: The incorrect manipulation of $\frac{P}{5-P} = \frac{P}{5} - 1$ or equivalent is awarded this dM0*.

Note: $(P) - (5-P) = e^{\frac{1}{3}t}\ \Rightarrow\ 2P - 5 = \frac{1}{3}t$ leading to $P = \ldots$ or equivalent is awarded this dM0*

(c)

B1: $1 + 4e^{-\frac{1}{3}t} > 1$ **and** $P < 5$ **and** a conclusion relating population (or even P) or meerkats to 5000.

For $P = \frac{25}{(5 + 20e^{-\frac{1}{3}t})}$, B1 can be awarded for $5 + 20e^{-\frac{1}{3}t} > 5$ **and** $P < 5$ **and** a conclusion relating population (or even P) or meerkats to 5000.

B1 can only be obtained if candidates have correct values of a and b in their $P = \frac{a}{(b + ce^{-\frac{1}{3}t})}$.

Award B0 for: As $t \to \infty$, $e^{-\frac{1}{3}t} \to 0$. So $P \to \frac{5}{(1+0)} = 5$, so population cannot exceed 5000,

unless the candidate also proves that $P = \frac{5}{(1 + 4e^{-\frac{1}{3}t})}$ oe. is an increasing function.

If unsure here, then send to review!

Alternative method for part (b)

B1M1*A1: as before for $\frac{1}{5}\ln P - \frac{1}{5}\ln(5-P) = \frac{1}{15}t\ \ (+c)$

Award 3rd M1for $\ln\left(\frac{P}{5-P}\right) = \frac{1}{3}t + c$

Award 4th M1 for $\frac{P}{5-P} = Ae^{\frac{1}{3}t}$

Award 2nd M1 for $t = 0, P = 1\ \Rightarrow\ \frac{1}{5-1} = Ae^0\ \ \left\{\Rightarrow\ A = \frac{1}{4}\right\}$

$\frac{P}{5-P} = \frac{1}{4}e^{\frac{1}{3}t}$

then award the final M1A1 in the same way.

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Write $\dfrac{1}{(H-5)(H+3)}$ in partial fraction form.

(3)

The depth of water in a storage tank is being monitored.

The depth of water in the tank, H metres, is modelled by the differential equation

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{(H-5)(H+3)}{40}$$

where t is the time, in days, from when monitoring began.

Given that the initial depth of water in the tank was 13 m,

(b) solve the differential equation to show that

$$H = \frac{10 + 3\,\mathrm{e}^{-0.2t}}{2 - \mathrm{e}^{-0.2t}}$$

(7)

(c) Hence find the time taken for the depth of water in the tank to fall to 8 m.

(3)

According to the model, the depth of water in the tank will eventually fall to k metres.

(d) State the value of the constant k.

(1)

14. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

(a) Write $\dfrac{1}{(H-5)(H+3)}$ in partial fraction form.

(3)

The depth of water in a storage tank is being monitored.
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According to the model, the depth of water in the tank will eventually fall to k metres.

(d) State the value of the constant k.

(1)

a) $\frac{A}{H-5} + \frac{B}{H+3} = A(H+3) + B(H-5)$

$A + B = 0$
$3A - 5B = 1$ $\Rightarrow$ $A = \frac{1}{8}$ $B = \frac{-1}{8}$

$\Rightarrow$ $\frac{1}{8(H-5)} + \frac{-1}{8(H+3)}$ //

b) $\frac{dH}{dt} = -\frac{(H+5)(H+3)}{40}$

$$\int \frac{1}{(H+5)(H+3)}\, dH = \int \frac{-1}{40}\, dt$$

$$\int \frac{1}{8(H-5)} + \frac{-1}{8(H+3)}\, dH = \frac{-1}{40} t + \ln K$$

$$\frac{1}{8}\ln(H-5) - \frac{1}{8}\ln(H+3) = \frac{-1}{40} t + \ln K$$

$$5 \ln\left[\frac{H-5}{H+3}\right] = -t + \ln \lambda$$

at $t=0$, $H=13$

$$5 \ln\left(\frac{8}{16}\right) = \ln \lambda$$

$$\lambda = -5\ln 2$$

$$\Rightarrow 5\ln\left[\frac{H-5}{H+3}\right] = -t - 5\ln 2$$

$$\ln\left|\frac{H-5}{H+3}\right| = -0.2t - \ln 2$$

$$\frac{H-5}{H+3} = e^{-0.2t-\ln 2} = \frac{e^{-0.2t}}{2}$$

$$\frac{H-5}{H+3} = \frac{e^{-0.2t}}{2}$$

$$2H-10 = e^{-0.2t}(H+3)$$

$$H(2-3e^{-0.2t}) = 3e^{-0.2t} + 10$$

$$H = \frac{10+3e^{-0.2t}}{2-3e^{-0.2t}}$$

//

$$H = \frac{10 + 3\,e^{-0.2t}}{2 - e^{-0.2t}}$$

(7)

(c) Hence find the time taken for the depth of water in the tank to fall to 8 m.

(3)

According to the model, the depth of water in the tank will eventually fall to k metres.

(d) State the value of the constant k.

(1)

c) $8 = \frac{10 + 3e^{-0.2t}}{2 - e^{-0.2t}}$

$16 - 8e^{-0.2t} = 10 + 3e^{-0.2t}$

$6 = 11e^{-0.2t}$

$\frac{6}{11} = e^{-0.2t}$

$t = -\frac{1}{5}\ln\left(\frac{6}{11}\right) = 3.03$ days

$t \to \infty$, $H \to 5$ //