#Exam Papers/2019/Further Maths

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SoHokMaths by A. Chan
AS Further Maths
Core Pure
2019 Full Paper

**Pearson Edexcel Level 3 GCE**

Centre Number

# Monday 13 May 2019

Afternoon (Time: 1 hour 40 minutes) | Paper Reference **8FM0-01**

# Further Mathematics

## Advanced Subsidiary
## Paper 1: Core Pure Mathematics

**Notional Component Grade Boundaries**

**Edexcel GCE AS/A Level (From 2015 and 2016)**

**June 2019**

**Further Mathematics: New Specification**

| AS notional component grade boundaries | | | Max Mark | A | B | C | D | E | U |
|---|---|---|---|---|---|---|---|---|---|
| 8FM0 | AS Further Mathematics Paper 01 | Raw | 80 | 49 | 41 | 33 | 25 | 18 | 0 |

1.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

**(2)**

The transformation $T$ of the plane is represented by the matrix **M**.

The triangle $R$ is transformed to the triangle $S$ by the transformation $T$.

Given that the area of $S$ is 63 square units,

(b) find the area of $R$.

**(2)**

(c) Show that the line $y = 2x$ is invariant under the transformation $T$.

**(2)**

(a) $M = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$  $\det M = 4(-7) - 2(-5)$

$= -28 + 10$

$= -18$

$\det M \neq 0$

NOT singular.

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(b)

Area scale factor $= 18$

$18\,(R) = 63$

$(R) = 3.5$

Area $\Delta$ $R = 3.5$ square units

---

**1.**

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

**(2)**

The transformation $T$ of the plane is represented by the matrix **M**.

The triangle $R$ is transformed to the triangle $S$ by the transformation $T$.

Given that the area of $S$ is 63 square units,

(b) find the area of $R$.

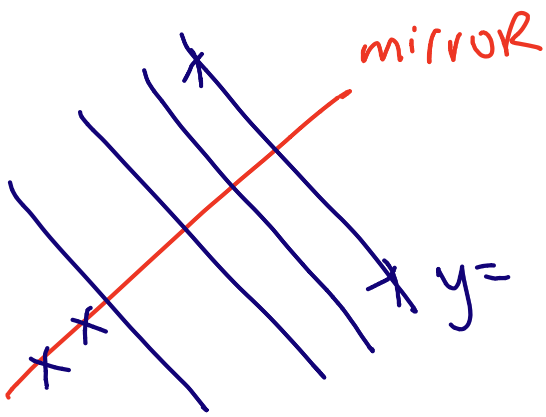
**(2)**

6+0

(c) Show that the line $y = 2x$ is invariant under the transformation $T$.

**(2)**

(c)




$y =$

$$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$x' = 4(x) - 5(2x) =$

$x' = 4x - 10x$

$x' = -6x$

$y' = 2x - 7(2x)$

$y' = -12x$

$y' = 2(x')$

$-12x = 2(-6x)$

$-12x = -12x$

LHS = RHS

$y = 2x$ is invariant //

**2.** The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

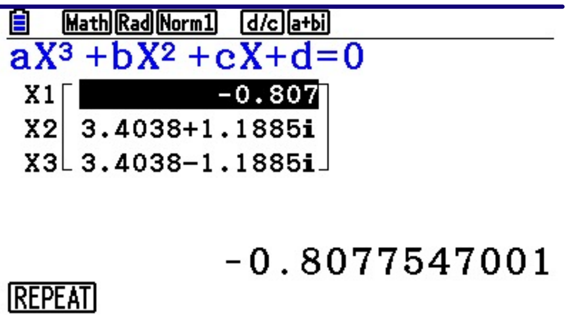
Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where $p$, $q$, $r$ and $s$ are integers to be found.

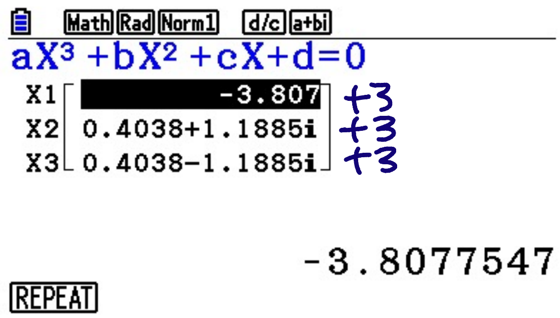
**(5)**

$\alpha$ is a root to $2x^3+6x^2-3x+12=0$

let $\alpha+3=w$

$\alpha=w-3$




$2\alpha^3+6\alpha^2-3\alpha+12=0$

$2(w-3)^3+6(w-3)^2-3(w-3)+12=0$

$2[w^3-3w^2(3^1)+3w(3^2)-3^3)+6(w^2-6w+9)$

$-3w+9+12=0$

$$\begin{aligned} &2w^3 - 18w^2 + 54w - 54 \\ &\quad + 6w^2 - 36w + 54 \qquad = 0 \\ &\qquad\quad - 3w + 21 \end{aligned}$$

$$2w^3 - 12w^2 + 15w + 21 = 0 \quad //$$

**3.** Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

**(6)**

1 .... — — k – k+1 – n

Basis

When $n=1$

$$LHS = \sum_{1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{(2-1)(2+1)} = \frac{1}{3}$$

$$RHS = \frac{1}{2+1} = \frac{1}{3}$$

true for $n=1$

---

Assumption

Assume true for $n=k$

$$\sum_{1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$$

---

Induction

for $n=k+1$

$$\boxed{\text{Aim: } \frac{k+1}{2(k+1)+1}}$$

$$LHS = \sum_{1}^{k+1} \frac{1}{(2r-1)(2r+1)}$$

$$= \sum_{1}^{k} \frac{1}{(2r-1)(2r+1)} + \sum_{k+1}^{k+1} \frac{1}{(2r-1)(2r+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

$$= \frac{k+1}{2(k+1)+1}$$

$$= RHS$$

LHS=RHS proved true for n=k+1

Proved true for n=1
Assumed true for n=k
Proved true for n=k+1
Therefore by Mathematical Induction,

that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

**4.** The line $l$ has equation

$$\frac{x+2}{1} = \frac{y-5}{-1} = \frac{z-4}{-3}$$

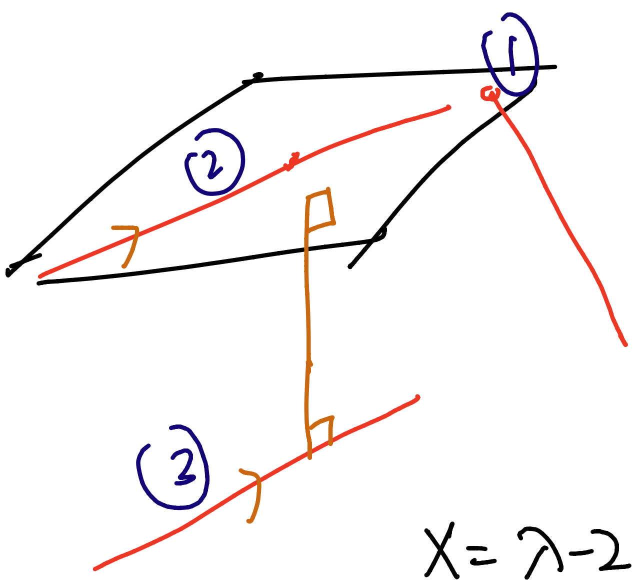
The plane $\Pi$ has equation

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -7$$

Determine whether the line $l$ intersects $\Pi$ at a single point, or lies in $\Pi$, or is parallel to $\Pi$ without intersecting it.

1 2 3

**(5)**




$$l: \frac{x+2}{1} = \frac{y-5}{-1} = \frac{z-4}{-3} = \lambda$$

$$x = \lambda - 2 \quad y = -\lambda + 5 \quad z = -3\lambda + 4$$

$$r \cdot n = a \cdot n$$

$$r \cdot \begin{pmatrix}1\\-2\\1\end{pmatrix} = -7$$

$$r = \begin{pmatrix}x\\y\\z\end{pmatrix} = \underbrace{\begin{pmatrix}-2\\5\\4\end{pmatrix}}_{\text{position}} + \lambda\underbrace{\begin{pmatrix}1\\-1\\-3\end{pmatrix}}_{\text{direction}}$$

---

test normal against direction of line

$$\begin{pmatrix}1\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-3\end{pmatrix} = 1(1) + -2(-1) + (1)(-3)$$

$$\boxed{a \cdot b = |a||b|\cos\theta} \qquad = 0 \; /\!/$$

normal $\perp$ to line

there for
$\Pi$ // line

position of line $= \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$
vector

$$\begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = LHS$$

$$= -2-10+4$$

$$= -8, \neq -7$$

point is not on plane.

$\Rightarrow$ parallel without intersecting.

**5.**

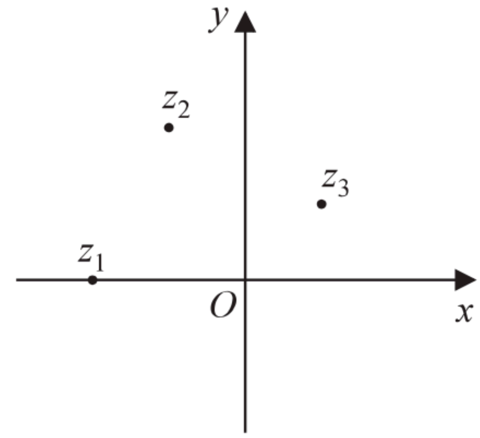



**Figure 1**

The complex numbers $z_1 = -2$, $z_2 = -1 + 2\mathrm{i}$ and $z_3 = 1 + \mathrm{i}$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + \mathrm{i}y$

(a) Explain why $z_1$, $z_2$ and $z_3$ cannot all be roots of a quartic polynomial equation with real coefficients.

**(2)**

(b) Show that $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \dfrac{\pi}{4}$

$\arg(a) - \arg(b)$

$= \arg\left(\frac{a}{b}\right)$

**(3)**

(c) Hence show that $\arctan(2) - \arctan\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$

**(2)**

A copy of Figure 1, labelled Diagram 1, is given on page 12.

(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leqslant |z - 1 - \mathrm{i}|$$

**(2)**

(a) $z_1 = -2$

$z_2 = -1 + 2i$

$z_3 = 1 + i$

roots appear in conjugates

$-2$ is real

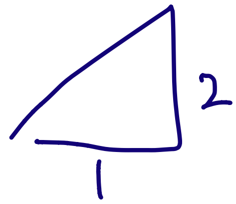
$-1+2i$ and $1+i$ are not

conjugates of the form

$a+bi$ and $a-bi$

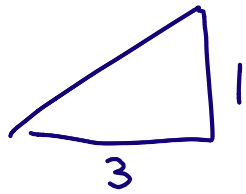
(b) $z_2 - z_1 = (-1+2i) - (-2)$

$= 1 + 2i$



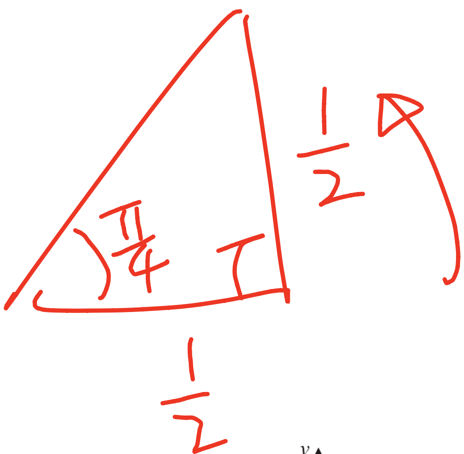

$z_3 - z_1 = (1+i) - (-2)$

$= 3 + i$




$\arg\left(\dfrac{1+2i}{3+i}\right) = \arg\left(\dfrac{1+2i}{3+i}\cdot\dfrac{3-i}{3-i}\right)$

$\arg\left(\dfrac{3+6i-i+2}{9+1}\right)$




$\arg\left(\dfrac{5+5i}{10}\right)$

$\arg\left(\frac{1}{2} + \frac{1}{2} i\right)$

$= \dfrac{\pi}{4}$

**5.**

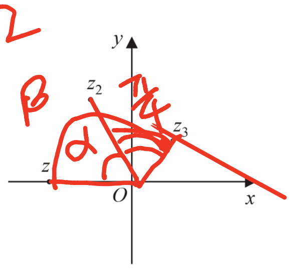



**Figure 1**

The complex numbers $z_1 = -2$, $z_2 = -1 + 2i$ and $z_3 = 1 + i$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + iy$

(a) Explain why $z_1$, $z_2$ and $z_3$ cannot all be roots of a quartic polynomial equation with real coefficients.

**(2)**

(b) Show that $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \dfrac{\pi}{4}$ $\qquad \arg(a) - \arg(b) = \arg\left(\frac{a}{b}\right)$

**(3)**

(c) Hence show that $\arctan(2) - \arctan\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$

**(2)**

A copy of Figure 1, labelled Diagram 1, is given on page 12.

(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leqslant |z - 1 - i|$$

**(2)**

$\arg(z_2 - z_1) = \tan^{-1}(2)$

$\arg(z_3 - z_1) = \tan^{-1}\left(\frac{1}{3}\right)$

$$\arctan 2 - \arctan \frac{1}{3}$$

$$= \arg(z_2 - z_1) - \arg(z_3 - z_1)$$

$$= \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right)$$

$$= \frac{\pi}{4}$$ as required //

(c)

$z_1 = -2$

$z_2 = -1 + 2i$

$z_3 = 1 + i$

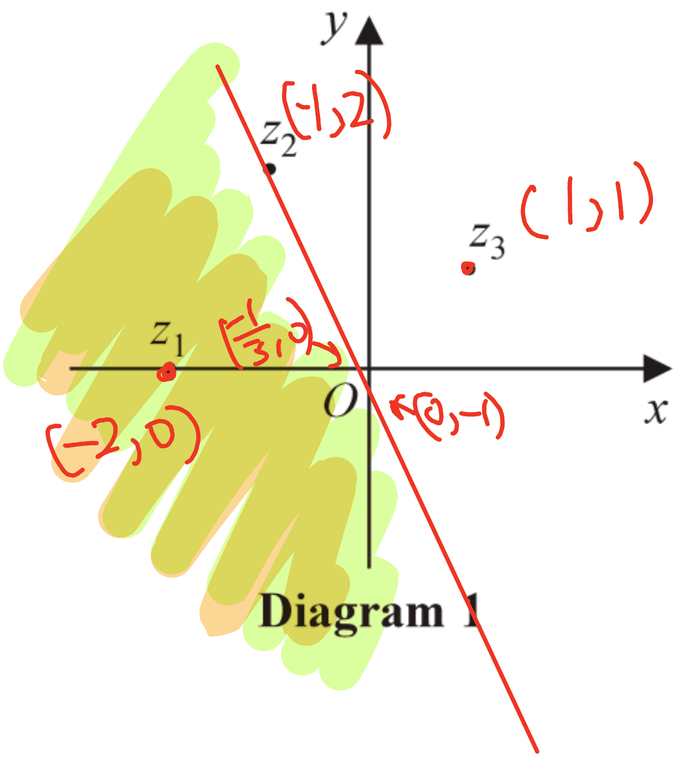



**Diagram 1**

$\frac{-2+1}{2}$ $\quad \frac{0+1}{2}$

$\boxed{-\frac{1}{2}, \frac{1}{2}}$

A copy of Figure 1, labelled Diagram 1, is given on page 12.

(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z+2| \leqslant |z-1-i|$$

**(2)**

distance from $z_1$ $\leq$ distance from $z_3$

$m = \frac{1-0}{1--2} = \frac{1}{3} \Rightarrow \frac{y - \frac{1}{2}}{x + \frac{1}{2}} = -3$

$$|x+iy+2| \le |x+iy-1-i|$$

$$\sqrt{\underbrace{(x+2)^2}+y^2} \le \sqrt{(x-1)^2+(y-1)^2}$$

$$x^2+4+4x+y^2 \le x^2-2x+1+y^2-2y+1$$

$$6x+4 \le -2y+2$$

$$2y \le -3x-2$$

$$y \le -3x-1 \quad //$$

$$y-\frac{1}{2} = -3x-\frac{3}{2}$$

$$y=-3x-1$$

$(0,-1)$ $\quad$ $3x=-1$

$x=-1/3$

**6.** An art display consists of an arrangement of $n$ marbles.

When arranged in ascending order of mass, the mass of the first marble is 10 grams. The mass of each subsequent marble is 3 grams more than the mass of the previous one, so that the $r$th marble has mass $(7 + 3r)$ grams.

(a) Show that the mean mass, in grams, of the marbles in the display is given by

$$\frac{1}{2}(3n+17)$$

$n = 85$

**(3)**

Given that there are 85 marbles in the display,

(b) use the standard summation formulae to find the standard deviation of the mass of the marbles in the display, giving your answer, in grams, to one decimal place.

**(6)**

Standard deviation $= \sqrt{\frac{S_{xx}}{n}}$ or $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$

sequence $\underbrace{10 \quad 13 \quad 16 \quad \ldots \quad 3r+7}_{n \text{ terms}}$

$$\sum_{1}^{n}(3r+7) = \text{total}$$

**Arithmetic series**

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$

$$3\sum_{1}^{n} r + 7n$$

$$\sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$3\left[\tfrac{1}{2}n(1+n)\right] + 7n$$

$$\frac{3(n)(n+1)}{2} + \frac{7n(2)}{2}$$

$$= \frac{1}{2}\left[3n^2 + 3n + 14n\right]$$

$$\text{Total} = \frac{1}{2}(3n^2 + 17n)$$

$$\text{mean} = \frac{1}{2}(3n+17) \;/\!/$$

(b) S.D. $\sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$

$\boxed{X = 3r+7}$

$$\frac{\sum x^2}{n} = \frac{\sum_{1}^{85} (3r+7)^2}{85}$$

$$= \frac{\sum_{1}^{85} [9r^2 + 42r + 49]}{85}$$

$n = 85$

$$= \frac{9\sum r^2 + 42\sum r + \sum 49}{85}$$

**Pure Mathematics**

**Summations**

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

$$= \frac{9\left[\frac{1}{6}n(n+1)(2n+1)\right] + 42\left[\frac{1}{2}n(n+1)\right] + 49n}{85}$$

$n = 85$

$$\Rightarrow \frac{2032690}{85} = \underline{\underline{23914}}$$

mean $= \frac{1}{2}(3n+17)$ //

$$\frac{\sum x}{n} = \frac{1}{2}(3(85)+17) = 136$$

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S.D. $\sqrt{23914 - 136^2} = 73.6$ (1dp)

grams //

**7.** $$f(z) = z^3 - 8z^2 + pz - 24$$

where $p$ is a real constant. $-$ $+$ $-$

Given that the equation $f(z) = 0$ has distinct roots

$$\alpha,\ \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $f(z) = 0$

**(6)**

(b) Hence find the value of $p$.

**(2)**

$$\alpha + \beta + \left(\alpha + \frac{12}{\alpha} - \beta\right) = \Sigma\alpha = 8 \quad (1)$$

$$2\alpha + \frac{12}{\alpha} = 8$$

$$2\alpha^2 + 12 = 8\alpha$$

$$\alpha\beta r = 24$$

$$\alpha^2 - 4\alpha + 6 = 0$$

$$\alpha\beta\left(\alpha + \frac{12}{\alpha} - \beta\right)$$

$$(\alpha - 2)^2 - 4 + 6 = 0$$

$$(\alpha - 2)^2 = -2$$

Math Rad Norm1 d/c a+bi

$aX^2 + bX + c = 0$

X1 2+1.4142i

X2 2−1.4142i

$2 + \sqrt{2}\,i$

REPEAT

$$\alpha = 2 \pm \sqrt{-2}$$

$$\alpha = 2 \pm \sqrt{2}\,i$$

$$z_1 = 2 + \sqrt{2}\,i$$
$$z_2 = 2 - \sqrt{2}\,i$$

$\boxed{z^2 - 4z + 6}$ is a factor

$$(z^3 - 8z^2 + pz - 24) = (z^2 - 4z + 6)(z - q)$$

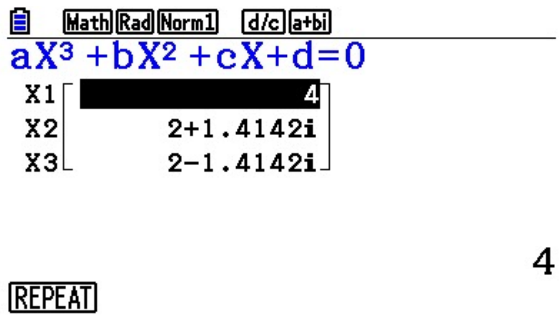
$$-24 = -6q$$

$$q = 4 \; //$$

$$z_1 = 2 + \sqrt{2}i$$

$$z_2 = 2 - \sqrt{2}i$$

$$z_3 = 4 \; //$$

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$$(z^2 - 4z + 6)(z - 4)$$

$$= \begin{array}{l} z^3 - 4z^2 + 6z \\ \quad -4z^2 + 16z - 24 \end{array}$$

$$= z^3 - 8z^2 + 22z - 24 \; //$$

$$p = 22 \; //$$

**8.** A gas company maintains a straight pipeline that passes under a mountain.

The pipeline is modelled as a straight line and one side of the mountain is modelled as a plane.

There are accessways from a control centre to two access points on the pipeline.

Modelling the control centre as the origin $O$, the two access points on the pipeline have coordinates $P(-300, 400, -150)$ and $Q(300, 300, -50)$, where the units are metres.

(a) Find a vector equation for the line $PQ$, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\lambda$ is a scalar parameter.

**(2)**

The equation of the plane modelling the side of the mountain is $2x + 3y - 5z = 300$

The company wants to create a new accessway from this side of the mountain to the pipeline.

The accessway will consist of a tunnel of shortest possible length between the pipeline and the point $M(100, k, 100)$ on this side of the mountain, where $k$ is a constant.

(b) Using the model, find

(i) the coordinates of the point at which this tunnel will meet the pipeline,

(ii) the length of this tunnel.

**(7)**

It is only practical to construct the new accessway if it will be significantly shorter than both of the existing accessways, $OP$ and $OQ$.

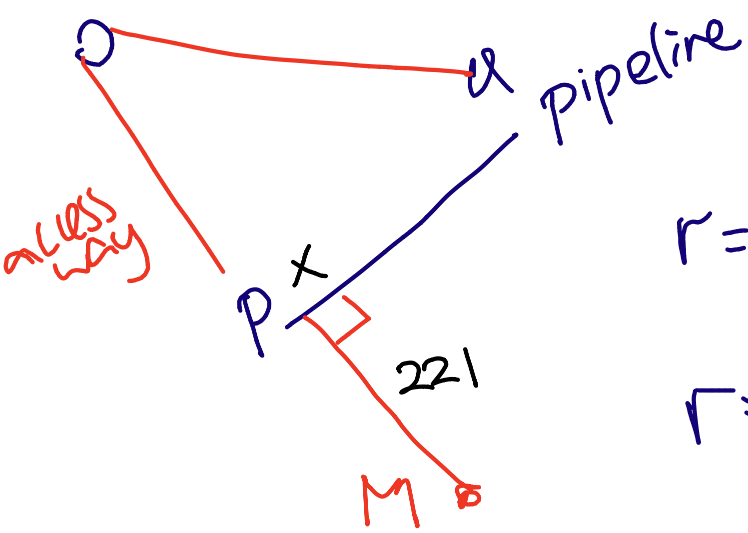
(c) Determine whether the company should build the new accessway.

**(2)**

(d) Suggest one limitation of the model.

**(1)**

might not be able to build a straight line from M to X, there might be obstacles.




$$r = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} -300-300 \\ 400-300 \\ -150--50 \end{pmatrix}$$

$$r = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} -600 \\ 100 \\ -100 \end{pmatrix}$$

$$r = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 1 \\ -1 \end{pmatrix}$$

M lie on the plane

sub $\begin{pmatrix} 100 \\ K \\ 100 \end{pmatrix}$ into $2x+3y-5z=300$

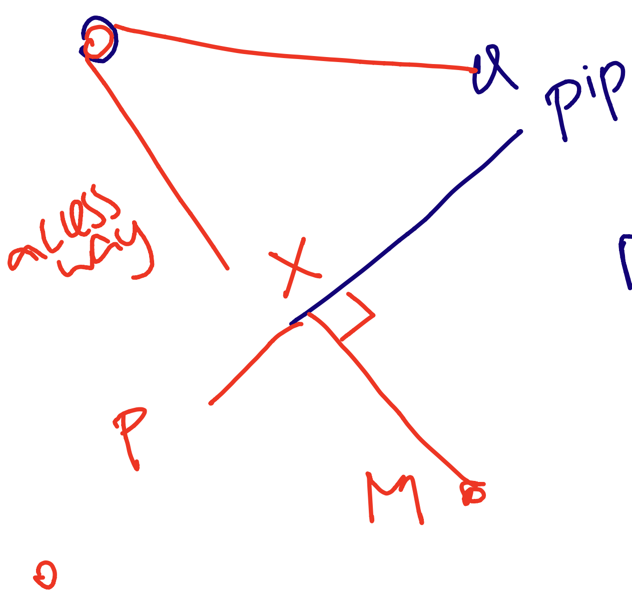
$$2(100)+3K-5(100)=300$$

$$3K = 300-200+500$$

$$3K = 600$$

$$K = 200$$

$$M = \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix}$$




$$r = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 1 \\ -1 \end{pmatrix}$$

$MP \perp PQ$

Let X to be the intersection,

(b) Using the model, find

(i) the coordinates of the point at which this tunnel will meet the pipeline,

(ii) the length of this tunnel.

**(7)**

$$\vec{MX} = \vec{MO} + \vec{OX}$$

PQ direction

$$= -\begin{pmatrix}100\\200\\100\end{pmatrix} + \begin{pmatrix}-300-6\lambda\\400+\lambda\\-150-\lambda\end{pmatrix} \qquad = \begin{pmatrix}-6\\1\\-1\end{pmatrix}$$

$$\vec{MX} = \begin{pmatrix}-400-6\lambda\\200+\lambda\\-250-\lambda\end{pmatrix}$$

Modelling the control centre as the origin $O$, the two access points on the pipeline have coordinates $P(-300, 400, -150)$ and $Q(300, 300, -50)$, where the units are metres.

$$\vec{MX} \cdot \vec{PQ} = 0$$

$$\begin{pmatrix}-400-6\lambda\\200+\lambda\\-250-\lambda\end{pmatrix} \cdot \begin{pmatrix}-6\\1\\-1\end{pmatrix} = 0$$

$$(-400-6\lambda)(-6) + 1(200+\lambda) - 1(-250-\lambda) = 0$$

$$\lambda = -75$$

$$\vec{OX} = \begin{pmatrix} -300-6\lambda \\ 400+\lambda \\ -150-\lambda \end{pmatrix} = \begin{pmatrix} 150 \\ 325 \\ -75 \end{pmatrix}$$

$$\vec{MX} = \begin{pmatrix} -400-6\lambda \\ 200+\lambda \\ -250-\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 50 \\ 125 \\ -175 \end{pmatrix}$$

$$|\vec{MX}| = \sqrt{50^2+125^2+175^2}$$

$$= 25\sqrt{78}$$

$$= 221 \ (3sf) \ //$$

metres //

It is only practical to construct the new accessway if it will be significantly shorter than both of the existing accessways, $OP$ and $OQ$.

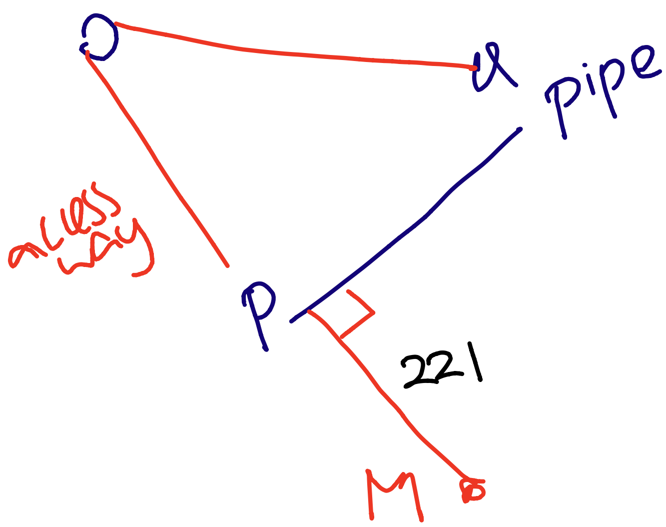
(c) Determine whether the company should build the new accessway.

**(2)**

(d) Suggest one limitation of the model.

**(1)**

Modelling the control centre as the origin $O$, the two access points on the pipeline have coordinates $P(-300, 400, -150)$ and $Q(300, 300, -50)$, where the units are metres.




$|OP|$

$= \sqrt{300^2+400^2+150^2}$

$= 50\sqrt{109}$

$= 522$ (3sf)

$\vec{OQ} = \sqrt{300^2+300^2+50^2}$

$= 50\sqrt{73}$

$= 427$ (3sf)

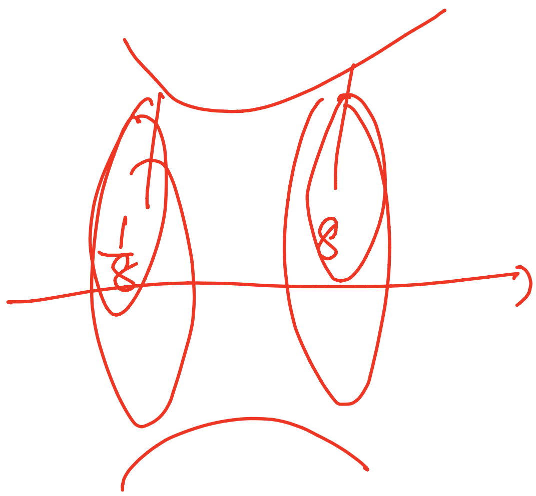
It is signifantly shorter, so should build the new accessway.

**9.** $$f(x) = 2x^{\frac{1}{3}} + x^{-\frac{2}{3}} \qquad x > 0$$

The finite region bounded by the curve $y = f(x)$, the line $x = \frac{1}{8}$, the $x$-axis and the line $x = 8$ is rotated through $\theta$ radians about the $x$-axis to form a solid of revolution.

Given that the volume of the solid formed is $\frac{461}{2}$ units cubed, use algebraic integration to find the angle $\theta$ through which the region is rotated.

**(8)**



$$\pi \int_{\frac{1}{8}}^{8} y^2 \, dx \Rightarrow \text{Full revolution}$$

$$\pi \int_{\frac{1}{8}}^{8} \left(2x^{\frac{1}{3}} + x^{\frac{-2}{3}}\right)^2 dx$$

$$\pi \int_{\frac{1}{8}}^{8} \left(4x^{\frac{2}{3}} + x^{\frac{-4}{3}} + 4x^{\frac{1}{3}}x^{\frac{-2}{3}}\right) dx$$

$$= \pi \int \left(4x^{\frac{2}{3}} + x^{\frac{-4}{3}} + 4x^{\frac{-1}{3}}\right) dx$$

$$= \pi \left[\frac{4x^{\frac{5}{3}}(3)}{5} + \frac{x^{\frac{-1}{3}}(3)}{-1} + \frac{4x^{\frac{2}{3}}(3)}{2}\right]_{\frac{1}{8}}^{8}$$

$$= \pi \left[ \left( \frac{12}{5}(8)^{\frac{5}{3}} - 3(8)^{\frac{1}{3}} + 6(8)^{\frac{2}{3}} \right) - \left( \frac{12}{5}\left(\frac{1}{8}\right)^{\frac{5}{3}} - 3\left(\frac{1}{8}\right)^{\frac{1}{3}} + 6\left(\frac{1}{8}\right)^{\frac{2}{3}} \right) \right]$$

$$= \pi [ 99.3 - -4.425 ]$$

$$= \pi [ 103.725 ]$$

---

$$\frac{461/2}{\pi \; 103.725} = \frac{\theta}{2\pi}$$

$$\theta = \frac{40}{9} \text{ radians}$$

Remain Juvenile

Reproduction Rate

$$J_{n+1} = aJ_n + 0.15 A_n$$

$$A_{n+1} = 0.08 J_n + 0.82 A_n$$

Grew old

Remain Adults

**10.** The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0, 1, 2, \dots$$

where $a$ is a constant, and $J_n$ and $A_n$ are the respective numbers of juvenile and adult chimpanzees $n$ years after the start of the study.

(a) Interpret the meaning of the constant $a$ in the context of the model.

multiplieR of Juvenile thct remain Juvenile **(1)**

Remain Juvenile

Reproduction Rate

$J_{n+1} = aJ_n + 0.15A_n$

$A_{n+1} = 0.08J_n + 0.82A_n$

Grew old

Remain Adults

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

$J_0 + A_0 = 64000$

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

$J_1 = 15360$

$A_1 = 43008$

(b) (i) Find, in terms of $a$

$$M^{-1} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

**(3)**

(ii) Hence, or otherwise, $\boxed{\text{find the value of } a}$

**(3)**

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

**(2)**

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix}$$

$\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = M \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}$

$M^{-1}\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = M^{-1} M \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}$

$M^{-1}\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}$ (bi)

$M = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}$

$\boxed{\frac{1}{\det M} C^T = M^{-1}}$

$\det M = \dfrac{1}{a(0.82) - 0.15(0.08)}$

$M^T = \begin{pmatrix} a & 0.08 \\ 0.15 & 0.82 \end{pmatrix}$

$C^T = \begin{pmatrix} +0.82 & -0.15 \\ -0.08 & +a \end{pmatrix}$

$M^{-1} = \dfrac{1}{0.82a - 0.012} \begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$

## bii

$$M^{-1}\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}$$

$J_0 + A_0 = 64000$
$J_1 = 15360$
$A_1 = 43008$

$$\frac{1}{0.82a - 0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}\begin{pmatrix} 15360 \\ 43008 \end{pmatrix} = \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}$$

$$0.82(15360) - 0.15(43008) = J_0(0.82a - 0.012)$$

$$-0.08(15360) + a(43008) = A_0(0.82a - 0.012)$$

$$\frac{0.82(15360) - 0.15(43008)}{0.82a - 0.012} + \frac{-0.08(15360) + a(43008)}{0.82a - 0.012}$$

$$J_0 \quad + \quad A_0 \quad = 64000$$

$$4915.2 + 43008a = 64000(0.82a - 0.012)$$

$$5683.2 = 9472a$$

$$\underline{a = 0.6}$$

otherwise

$$M\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} J_1 \\ A_1 \end{pmatrix}$$

$$\begin{cases} ax + 0.15y = 15360 \\ 0.08x + 0.82y = 43008 \\ x + y = 64000 \end{cases}$$

an X+bn Y=Cn
X [12800]
Y [51200]

12800

REPEAT

$\therefore a = 0.6$ //

## biii

$$M^{-1} = \frac{1}{0.82a - 0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$$

$$M^{-1} = \frac{25}{12}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & 0.6 \end{pmatrix}$$

$$0.82(15360) - 0.15(43008) = J_0(0.82a - 0.012)$$

$$J_0 = 12800$$

$$-0.08(15360) + a(43008) = A_0(0.82a - 0.012)$$

$$A_0 = 51200$$

$J_0 + A_0 = 64000$
$J_1 = 15360$
$A_1 = 43008$

$$J_1 - J_0 = 15360 - 12800$$

$$= \underline{+2560}$$

15360−12800
2560

$$\begin{bmatrix} x & 0.15 \\ 0.08 & 0.82 \end{bmatrix}\begin{bmatrix} 12800 \\ 51200 \end{bmatrix}$$

$$\begin{bmatrix} 15360 \\ 43008 \end{bmatrix}$$

2×2 | 3×3 | m×n | 2×1 | 3×1

[43008]

$$\begin{bmatrix} x & 0.15 \\ 0.08 & 0.82 \end{bmatrix}^{-1}\begin{bmatrix} 15360 \\ 43008 \end{bmatrix}$$

$$\begin{bmatrix} 12800 \\ 51200 \end{bmatrix}$$

2×2 | 3×3 | m×n | 2×1 | 3×1

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

NOT suitable since $J_1 > J_0$

**(1)**

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

**(2)**

Remain Juvenile

Reproduction Rate

$$J_{n+1} = aJ_n + 0.15A_n$$
$$A_{n+1} = 0.08J_n + 0.82A_n$$

Grew old

Remain Adults

(d)

Let V to be "very old chimpanzee"

$$J_{n+1} = aJ_n + 0.15A_n + 0V_n$$
$$A_{n+1} = 0.08J_n + bA_n + 0V_n$$
$$V_{n+1} = 0J_n + cA_n + dV_n$$

introduce a third variable, such that

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 & 0 \\ 0.08 & b & 0 \\ 0 & c & d \end{pmatrix} \begin{pmatrix} J_n \\ A_n \\ V_{n+1} \end{pmatrix}$$

$b + c = 0.82$

$d \Rightarrow$ remain very old % //