STEP 1 Mathematics 2019 Question 1 Coordinate Geometry & Differentiation

1 A straight line passes through the fixed point $(1\,,\,k)$ and has gradient $-\tan\theta$, where $k > 0$ and $0 < \theta < \frac{1}{2}\pi$. Find, in terms of θ and k, the coordinates of the points X and Y where the line meets the x-axis and the y-axis respectively.

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$$\frac{1 - \cos\alpha}{1 - \sin\alpha} = k.$$

Find and simplify an expression for the minimum value of L in terms of α.

Explanation of Results STEP 2019

All STEP questions are marked out of 20. The mark scheme for each question is designed to reward candidates who make good progress towards a solution. A candidate reaching the correct answer will receive full marks, regardless of the method used to answer the question.

All the questions that are attempted by a student are marked. However, only the 6 best answers are used in the calculation of the final grade for the paper.

There are five grades for STEP Mathematics which are:
S – Outstanding
1 – Very Good
2 – Good
3 – Satisfactory
U – Unclassified

3 hours 6 Q

The rest of this document presents, for each paper, the grade boundaries (minimum scores required to achieve each grade), cumulative percentage of candidates achieving each grade, and a graph showing the score distribution (percentage of candidates on each mark).

STEP Mathematics I (9465)

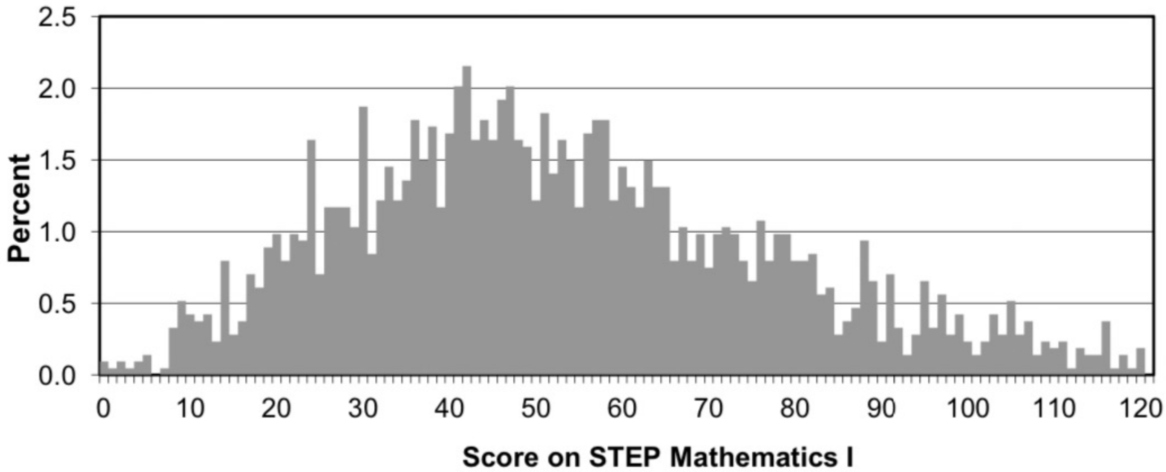
Grade boundaries

Maximum Mark	S	1	2	3	U
120	105	76	52	30	0

Cumulative percentage achieving each grade

Maximum Mark	S	1	2	3	U
120	3.3	18.7	47.7	82.9	100.0

Distribution of scores



Cambridge Assessment Admissions Testing

Sixth Term Examination Papers **9465**
MATHEMATICS 1 Morning
Monday 10 June 2019 Time: 3 hours

5417310365

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

Make sure you fill in page 1 **AND** page 3 of the answer booklet with your details.

INFORMATION FOR CANDIDATES

There are 11 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

There is NO Mathematical Formulae Booklet.

Calculators are not permitted.

Wait to be told you may begin before turning this page.

This question paper consists of 7 printed pages and 5 blank pages.

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Section A: Pure Mathematics

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$$\frac{1-\cos\alpha}{1-\sin\alpha}=k.$$

Find and simplify an expression for the minimum value of L in terms of α.

2 The curve C is given parametrically by the equations $x=3t^2$, $y=2t^3$. Show that the equation of the tangent to C at the point $(3p^2\,,\;2p^3)$ is $y=px-p^3$.

Find the point of intersection of the tangents to C at the distinct points $(3p^2\,,\;2p^3)$ and $(3q^2\,,\;2q^3)$. Hence show that, if these two tangents are perpendicular, their point of intersection is $(u^2+1\,,\,-u)$, where $u=p+q$.

The curve $\widetilde{C}$ is given parametrically by the equations $x=u^2+1$, $y=-u$. Find the coordinates of the points that lie on both C and $\widetilde{C}$.

Sketch C and $\widetilde{C}$ on the same axes.

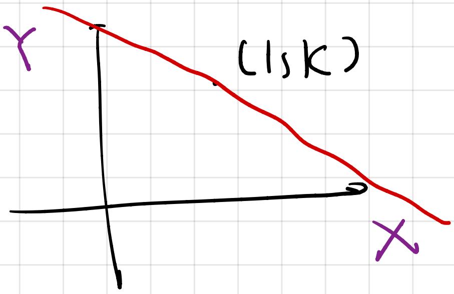
3 By first multiplying the numerator and the denominator of the integrand by $(1-\sin x)$, evaluate

$$\int_0^{\frac{1}{4}\pi}\frac{1}{1+\sin x}\,\mathrm{d}x\,.$$

Evaluate also:

$$\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi}\frac{1}{1+\sec x}\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{1}{3}\pi}\frac{1}{(1+\sin x)^2}\,\mathrm{d}x\,.$$

STEP 1 Mathematics 2019 Question 1 Coordinate Geometry & Differentiation



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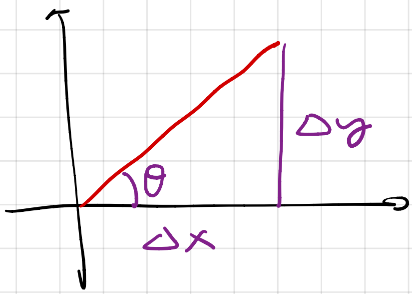
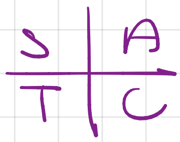


$\frac{\Delta y}{\Delta x} = \tan\theta$

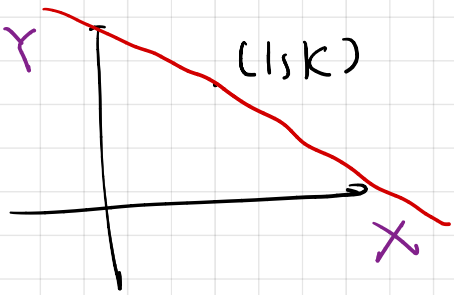


$\tan\theta > 0$

$y-k = -\tan\theta\,(x-1)$ — line.

when $x=0$, $y-k = -\tan\theta\,(-1)$

$y = \tan\theta + k$



$Y: (0, \tan\theta + k)$

when $y = 0$,

$0 - k = -\tan\theta(x-1)$

$-k = -\tan\theta\, x + \tan\theta$

$\tan\theta\, x = \tan\theta + k$

$x = \dfrac{\tan\theta + k}{\tan\theta}$

$x = 1 + \dfrac{k}{\tan\theta}$

$X: \left(1 + \dfrac{k}{\tan\theta}, 0\right)$

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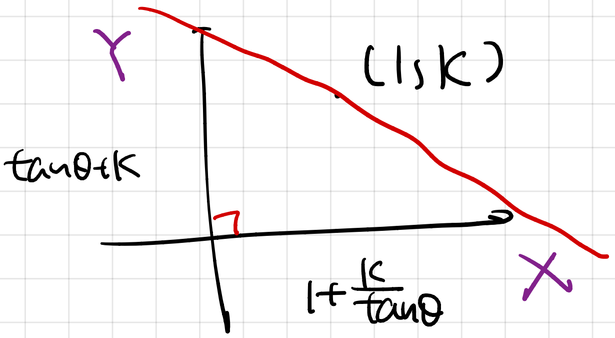
$X.\ \left(1+\frac{k}{\tan\theta}\ ,\ 0\right)$

$Y:\ \left(\ 0\ ,\ \tan\theta+k\right)$



$$A = \left(1+\frac{k}{\tan\theta}\right)\cdot\left(\tan\theta+k\right)\cdot\frac{1}{2}$$

$$A = \left(\tan\theta + k + k + \frac{k^2}{\tan\theta}\right)\frac{1}{2}$$

$$A = \left(\tan\theta + 2k + k^2\cot\theta\right)\frac{1}{2}$$

$$\frac{dA}{d\theta} = \frac{1}{2}\left(\sec^2\theta - k^2\operatorname{cosec}^2\theta\right)$$

$$\frac{dA}{d\theta} = 0 \qquad\qquad \underline{\tan^2\theta + 1 = \sec^2\theta}$$

$$\frac{\sec^2\theta - k^2\operatorname{cosec}^2\theta}{2} = 0$$

$$\frac{1}{\cos^2\theta} - \frac{k^2}{\sin^2\theta} = 0$$

$$\frac{1}{\cos^2\theta} = \frac{k^2}{\sin^2\theta}$$

$$\sin^2\theta = \cos^2\theta\, k^2$$

$$1 - \cos^2\theta = \cos^2\theta\, k^2$$

$$1 = (k^2+1)\cos^2\theta$$

$$\cos^2\theta = \frac{1}{k^2+1}$$

$$\sec^2\theta = k^2+1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan^2\theta = k^2+1-1$$

$$\tan^2\theta = k^2$$

$$\tan\theta = \pm k$$

$$\tan\theta = k \qquad \left(0 < \theta < \frac{\pi}{2} \right)$$

$$A = \left(\tan\theta + k + k + \frac{k^2}{\tan\theta} \right) \frac{1}{2}$$

$$A = \left(3k + \frac{k^2}{k} \right) \frac{1}{2}$$

$$\underline{\underline{A = 2K}}$$

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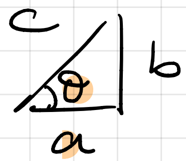
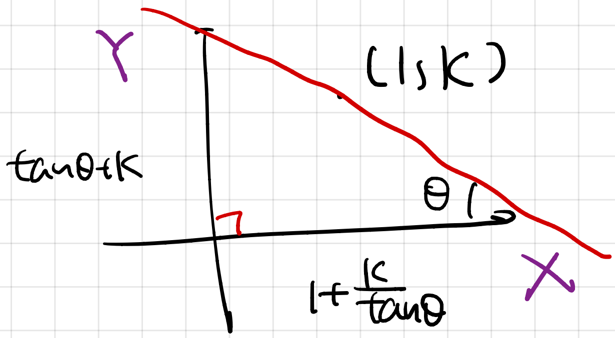


S | A
T | C

X: $\left(1 + \frac{k}{\tan\theta}, 0\right)$

Y: $(0, \tan\theta + k)$



$\frac{b}{c} = \sin\theta$

$\frac{b}{\sin\theta} = c$

$$\frac{\tan\theta + k}{XY} = \sin\theta$$

$$XY = \frac{\tan\theta + k}{\sin\theta}$$

$$L = \tan\theta + k + 1 + \frac{k}{\tan\theta} + \frac{\tan\theta + k}{\sin\theta}$$

$$L = 1 + \tan\theta + k + k\cot\theta + \frac{1}{\cos\theta} + k\operatorname{cosec}\theta$$

$$L = 1 + \tan\theta + \sec\theta + k(1 + \cot\theta + \operatorname{cosec}\theta)$$

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$$\frac{dL}{d\theta} = \sec^2\theta + \tan\theta\sec\theta + k(-\operatorname{cosec}^2\theta - \cot\theta\operatorname{cosec}\theta)$$

$$\frac{dL}{d\theta} = 0 \qquad \theta = \alpha.$$

$$k(\operatorname{cosec}^2\theta + \cot\theta\operatorname{cosec}\theta) = \sec^2\theta + \tan\theta\sec\theta$$

$$k\left(\frac{1}{\sin^2\theta} + \frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta}\right) = \left(\frac{1}{\cos^2\theta} + \frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta}\right)$$

$$k\left(\frac{1+\cos\theta}{\sin^2\theta}\right) = \left(\frac{1+\sin\theta}{\cos^2\theta}\right)$$

$\dfrac{1-\cos\theta}{1-\sin\theta}$

$$K = \left(\frac{1+\sin\theta}{\cos^2\theta}\right)\cdot\left(\frac{\sin^2\theta}{1+\cos\theta}\right)$$

$$K = \frac{(1+\sin\theta)(1-\cos^2\theta)}{(1+\cos\theta)(1-\sin^2\theta)}$$

$$K = \frac{\cancel{(1+\sin\theta)}(1-\cos\theta)\cancel{(1+\cos\theta)}}{\cancel{(1+\cos\theta)}(1-\sin\theta)\cancel{(1+\sin\theta)}}$$

$$K = \frac{1-\cos\theta}{1-\sin\theta} \quad \text{as required.}$$

$\theta = \alpha$, $\quad K = \dfrac{1-\cos\alpha}{1-\sin\alpha}$

$\theta = \alpha.$

$$L = 1+\tan\theta+\sec\theta + K\left(1+\cot\theta+\operatorname{cosec}\theta\right)$$

$$L = 1+\tan\alpha+\sec\alpha+\frac{1-\cos\alpha}{1-\sin\alpha}\left(1+\cot\alpha+\operatorname{cosec}\alpha\right)$$

$$L = 1+\frac{\sin\alpha}{\cos\alpha}+\frac{1}{\cos\alpha}+\frac{1-\cos\alpha}{1-\sin\alpha}\left(1+\frac{\cos\alpha}{\sin\alpha}+\frac{1}{\sin\alpha}\right)$$

$$L = \frac{\sin\alpha+\cos\alpha+1}{\cos\alpha}+\frac{1-\cos\alpha}{1-\sin\alpha}\left(\frac{\sin\alpha+\cos\alpha+1}{\sin\alpha}\right)$$

$$L = \frac{(\sin\alpha+\cos\alpha+1)}{\cos\alpha} + \frac{(1-\cos\alpha)(\sin\alpha+\cos\alpha+1)}{\sin\alpha\,(1-\sin\alpha)}$$

$$L = \frac{(\sin\alpha+\cos\alpha+1)\left(\sin\alpha(1-\sin\alpha)+(1-\cos\alpha)\cos\alpha\right)}{\sin\alpha\cos\alpha\,(1-\sin\alpha)}$$

$$L = \frac{(\sin\alpha+\cos\alpha+1)\left[\sin\alpha-\sin^2\alpha+\cos\alpha-\cos^2\alpha\right]}{\sin\alpha\cos\alpha\,(1-\sin\alpha)}$$

$$L = \frac{(\sin\alpha+\cos\alpha+1)(\sin\alpha+\cos\alpha-1)}{\sin\alpha\cos\alpha\,(1-\sin\alpha)}$$

$$L = \frac{(\sin\alpha+\cos\alpha)^2-1}{\sin\alpha\cos\alpha-\sin^2\alpha\cos\alpha}$$

$$L = \frac{\cancel{\sin^2\alpha+\cos^2\alpha}+2\sin\alpha\cancel{\cos\alpha}\cancel{-1}}{\sin\alpha\cancel{\cos\alpha}\,(1-\sin\alpha)}$$

$$L = \frac{2}{1-\sin\alpha}$$

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